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On vortex sound at low Mach number

By W. MÖHRING
Max-Planck-Institut für Strömungsforschung, Göttingen, West Germany

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A transformation is described which relates the sound generated by low Mach number flow to the flow vorticity. For compact flow fields the apparent sound source is of quadrupole type and linear in the vorticity and therefore also linear in the flow velocity. This scheme is applied to the sound generated by the interaction of two identical thin vortex rings. Then a flow field with a number of compact vortices is discussed. It is found that each vortex can be replaced acoustically by a dipole related to the impulse of the vortex, plus the quadrupole just mentioned plus a spherically symmetric sound source related to the energy of the vortex. An application to low Mach number free-space turbulence shows that the generated sound is related to the vorticity correlation tensor.

1. Introduction

Different methods have been proposed for calculating the sound generated by low Mach number flow. In these methods an inhomogeneous wave equation of the form

$$a_0^{-2}\,\partial^2 p/\partial t^2 - \Delta p = q \tag{1}$$

is derived from the compressible hydrodynamic equations. In (1), a_0 denotes the ambient speed of sound, p a quantity which agrees in the far field with the acoustic pressure fluctuations and q a source term which can be calculated from the flow, which is assumed to be incompressible and known. Different expressions for q have been derived which can be shown to agree in the pressure far field obtained from (1). Lighthill (1952) showed that q can be written as a quadrupole source

$$q = \partial^2 T_{ij}/\partial x^i\,\partial x^j, \quad T_{ij} = \rho_0 u^i u^j, \tag{2}$$

where ρ_0 is the ambient density and the u^i are the flow velocity components. Ribner (1962) showed that q can be chosen as

$$q = a_0^{-2} p_{tt}^{\mathrm{inc}}, \quad \Delta p^{\mathrm{inc}} = -\,\partial^2 T_{ij}/\partial x^i\,\partial x^j. \tag{3}$$

In this case the incompressible pressure acts as the sound source. A third type of sound source has been derived by Powell (1964) and Howe (1975), who showed that q can be taken as

$$q = \rho_0 \operatorname{div} \mathbf{L}, \quad \mathbf{L} = \mathbf{w} \times \mathbf{u}, \tag{4}$$

where $\mathbf{w} = \operatorname{curl} \mathbf{u}$ denotes the vorticity vector. One observes that (3) expresses the generated sound in terms of monopole sources, (4) expresses it in terms of dipole sources and (2) expresses it in terms of quadrupole sources. As these sources lead to the same sound field, the total monopole strength and dipole strength of the sources in (3) and (4) vanish. The main difference in these different sources, apart from their different types, consists of their degree of compactness. For the often-treated case of an unsteady flow which has a quadrupole-type far-field behaviour (this occurs if there are no mass sources and no forces in the flow region), the Lighthill quadrupole source decays like $|\mathbf{x}|^{-10}$ and Ribner's monopoles as $|\mathbf{x}|^{-3}$, therefore the total monopole strength leads in this case to a divergent integral. If the flow is generated by vortices which are confined to a finite region, Powell's sources defined in (4) vanish outside this region. These sources also show that the sound is generated by vortices. In infinite space there is no sound without vortices. Equation (4) does not, however, represent the sound in terms of the vorticity alone: the flow velocity occurs also. This leads to a difficulty if one tries to apply the source term of (4) to the sound generated by vortex filaments. Then $\mathbf{w}$ shows a δ-function-like behaviour and $\mathbf{u}$ is singular at the singularity of the δ-function. This difficulty is usually overcome by substituting for $\mathbf{u}$ the velocity of the filament. A similar difficulty prevents the application of (4) to vortex sheets.

One also notices that (1) and the source terms defined in (2)–(4) are local relations. The derivatives of p at a given position are related to derivatives of the flow velocity at that position. In (3) the source is locally related to the incompressible pressure, which is locally related to the velocity components. Furthermore, one observes that the source terms are quadratic functions of the flow velocities, which implies that the same is true of the acoustic pressure. This can be clearly seen from a Green's function representation

$$p(\mathbf{x},t) = \int G(\mathbf{x},\mathbf{y},t-t')\,q(\mathbf{y},t')\,d^3y\,dt', \tag{5}$$

where G denotes the Green's function for (1) with the appropriate boundary conditions. One finds from (5) that the acoustic intensity

$$I = \frac{p^2(\mathbf{x},t)}{\rho_0 a_0} = \frac{1}{\rho_0 a_0}\iint G(\mathbf{x},\mathbf{y},t-t')\,G(\mathbf{x},\mathbf{y}',t-t'')\,q(\mathbf{y},t')\,q(\mathbf{y}',t'')\,d^3y\,d^3y'\,dt'\,dt'' \tag{6}$$

is related to expressions quadratic in q and therefore to expressions biquadratic in the flow velocities. From (6) one obtains the well-known relations between the sound generated by low Mach number turbulence and the fourth-order velocity correlations.

2. Basic equations

It is well known that the hydrodynamic equations are nonlinear. For this reason there are relations between quantities linear and quadratic in the velocity, e.g. Helmholtz's vortex equation

$$\partial\mathbf{w}/\partial t + \operatorname{curl}\mathbf{L} = 0, \tag{7}$$

which is valid for incompressible flow. To achieve a representation of the acoustic pressure in terms of curl $\mathbf{L}$ an integration by parts of (5) with the source term of (4) is performed:

$$p(\mathbf{x},t) = -\rho_0\int \nabla_y G(\mathbf{x},\mathbf{y},t-t')\,.\,\mathbf{L}(\mathbf{y},t')\,d^3y\,dt'. \tag{8}$$

Now one could apply (7) if one were able to find a vector Green's function which obeys the equation

$$\nabla_y G = \nabla_y \times \mathbf{G}. \tag{9}$$

Then one would find from (8)

$$p = -\rho_0\int \mathbf{G}\,.\,\nabla_y \times \mathbf{L}\,d^3y\,dt$$

and from Helmholtz's vortex equation (7)

$$p = \rho_0 \int \mathbf{G}(\mathbf{x}, \mathbf{y}, t-t') . \frac{\partial \mathbf{w}(\mathbf{y}, t')}{\partial t'} d^3y\, dt'$$

$$= \rho_0 \frac{\partial}{\partial t} \int \mathbf{G}(\mathbf{x}, \mathbf{y}, t-t') . \mathbf{w}(\mathbf{y}, t')\, d^3y\, dt'. \tag{10}$$

Equation (10) differs in several respects from (5) with the source from (2), (3) or (4). It is derived not from differential equations but from integral representations, and therefore probably does not represent a local relation. The integration region in (10) is the region of non-vanishing vorticity, just as for Powell's sources. Furthermore, the integrand in (10) does not contain the flow velocity, therefore the acoustic pressure can be calculated from the vorticity of the flow alone. For this reason it can easily be applied to the sound generated by moving vortex filaments and vortex sheets.

Another marked difference between (10) and (5) is contained in the fact that the pressure depends linearly on the vorticity field and therefore also linearly on the flow velocity. This means that all vortices contribute to the generated sound and that these contributions add linearly. The influence of their interaction enters (10) only through their motion. According to (6) the sound intensity derived from (10) depends on the second-order vorticity correlations, which correspond to derivatives of the second-order velocity correlations instead of the fourth-order correlations in (6).

The main difficulty with (10) consists of solving (9). Although (9) represents a purely acoustic problem, completely independent of the flow field which generates the sound, one finds that (9) is in general unsolvable. It consists of three first-order partial differential equations, whose integrability condition, which is obtained by applying the divergence operator to (9), is

$$\Delta_y G(\mathbf{x}, \mathbf{y}, t-t') = 0, \tag{11}$$

a condition which is usually not satisfied by the wave-equation Green's function. In cases where G is a symmetric function of its arguments it satisfies instead the inhomogeneous wave equation

$$\Delta_y G(\mathbf{x}, \mathbf{y}, t-t') = a_0^{-2}\, \partial^2 G/\partial t'^2 - \delta(\mathbf{x}-\mathbf{y})\, \delta(t-t'). \tag{12}$$

Therefore a vector function $\mathbf{G}$ exists only if the right-hand side of (12) vanishes. The second term vanishes for $\mathbf{x} \neq \mathbf{y}$, so this term vanishes if it is assumed that $\mathbf{x}$ is in the wave region and $\mathbf{y}$ in the flow region, which is usually the interesting case. The first term on the right-hand side contains an a_0^{-2} factor, therefore it is small for large values of the speed of sound and might perhaps vanish to lowest order in the Mach number. Examples will clarify this. Notice that the differentiation refers to the variable $\mathbf{y}$, the integration variable which is restricted to the flow region.

3. Examples

As a first example a line vortex with circulation Γ near a semi-infinite rigid plane is considered, a problem treated by Crighton (1972) and Howe (1975) (figure 1). The appropriate approximate Green's function, valid for $\mathbf{x}$ in the wave region and $\mathbf{y}$ in the

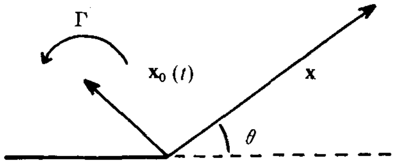

FIGURE 1. Vortex near a semi-infinite rigid plane.

flow region, for low Mach number flow has been determined by Howe (1975). It is given by

$$G(\mathbf{x},\mathbf{y},t-t') = \frac{\phi(\mathbf{y})\,\phi(\mathbf{x})}{\pi x}\,\delta\left(t-t'-\frac{x}{a_0}\right), \tag{13}$$

where $|\mathbf{x}| = x$ and $\phi(\mathbf{x}) = x^{\frac{1}{2}}\sin\frac{1}{2}\theta$ is the potential function which describes irrotational incompressible flow around a half-plane. This function satisfies the solvability condition (11), therefore a vector function $\mathbf{G}$ can be determined. One easily finds

$$\mathbf{G}(\mathbf{x},\mathbf{y},t-t') = \psi(\mathbf{y})\frac{\phi(\mathbf{x})}{\pi \mathbf{x}}\,\delta\left(t-t'-\frac{x}{a_0}\right)\mathbf{k},$$

where $\psi(\mathbf{y})$ is the stream function conjugate to $\phi(\mathbf{y})$ and $\mathbf{k}$ is a unit vector in the z direction. If the position of the vortex at time t' is denoted by $\mathbf{x}_0(t')$, the flow vorticity is given by $\mathbf{w}(\mathbf{y},t') = \Gamma\delta(\mathbf{y}-\mathbf{x}_0(t'))\,\mathbf{k}$. Then (10) leads to

$$p = \rho_0\frac{\phi(\mathbf{x})}{\pi x}\frac{\partial}{\partial t}\psi\left(\mathbf{x}_0\left(t-\frac{x}{a_0}\right)\right),$$

which is exactly Crighton's result in Howe's form.

The next example refers to an unsteady low Mach number flow in free space. For large values of x in the wave region, the Green's function is given by

$$G(\mathbf{x},\mathbf{y},t-t') = \frac{1}{4\pi x}\,\delta\left(t-t'-\frac{x}{a_0}+\frac{\mathbf{x}\,.\,\mathbf{y}}{x a_0}\right), \tag{14}$$

which can be approximated as

$$G(\mathbf{x},\mathbf{y},t-t') = \frac{1}{4\pi x}\left(\delta(t^r)+\frac{\mathbf{x}\,.\,\mathbf{y}}{x a_0}\delta'(t^r)+\frac{(\mathbf{x}\,.\,\mathbf{y})^2}{2x^2a_0^2}\delta''(t^r)+\ldots\right), \tag{15}$$

$$t^r = t-t'-x/a_0.$$

If this is inserted in (8), one finds that the contributions from the first two terms of the right-hand side of (15) vanish because $\int \mathbf{L}\,d^3y = 0$, as has been shown by Powell (1964, equation (42). So we restrict ourselves to the third term of (15). This term does not satisfy the solvability condition (11), therefore it might seem that a $\mathbf{G}$ function does not exist. However, if one observes that $\mathbf{G}$ is to be applied not to an arbitrary function, but to a function which obeys certain conditions, a suitable $\mathbf{G}$ can be determined. One starts from the identity

$$\nabla_y(\mathbf{x}\,.\,\mathbf{y})^2 = \tfrac{2}{3}\{\nabla_y\times[(\mathbf{x}\,.\,\mathbf{y})\,\mathbf{x}\times\mathbf{y}]+x^2\mathbf{y}\}$$

and finds

$$\int\nabla_y(\mathbf{x}\,.\,\mathbf{y})^2\,.\,\mathbf{L}d^3y = \tfrac{2}{3}\int\nabla_y\times[(\mathbf{x}\,.\,\mathbf{y})\,\mathbf{x}\times\mathbf{y}]\,.\,\mathbf{L}\,d^3y+\tfrac{2}{3}x^2\int\mathbf{y}\,.\,\mathbf{L}\,d^3y. \tag{16}$$

The second term on the right-hand side of this equation is constant because of Powell's 'three-sound-pressures theorem'. It represents the total hydrodynamic energy in the

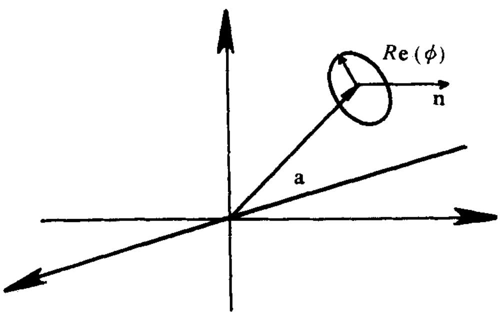

FIGURE 2. The ring vortex.

flow (Powell 1964, equation 73), therefore it can be ignored in (8) and **G** can be chosen as

$$\mathbf{G} = \frac{1}{12\pi a_0^2 x^3}\delta''\left(t-t'-\frac{x}{a_0}\right)(\mathbf{x}.\mathbf{y})\,\mathbf{x}\times\mathbf{y}.$$

Then (10) leads to

$$p = \frac{\rho_0}{12\pi a_0^2 x^3}\frac{\partial^3}{\partial t^3}\int(\mathbf{x}.\mathbf{y})\,\mathbf{y}.(\mathbf{w}\times\mathbf{x})\,d^3y, \quad (17)$$

where the retarded time $t' = t-x/a_0$ is to be used in **w**.

One notices that only those components of **w** which are orthogonal to the vector to the observation point contribute to the integral. In this approximation a vortex element does not radiate in the direction of its axis.

The integral in (17) can be easily evaluated for a circular vortex ring of radius R (figure 2). If it is situated at $\mathbf{y}_w = \mathbf{a}+R\mathbf{e}(\phi)$, $0 \leqslant \phi \leqslant 2\pi$, where **a** is a vector to the centre of the ring and $\mathbf{e}(\phi)$ denotes a unit vector in the vortex ring's plane, the vorticity s given by $\mathbf{w} = \Gamma\mathbf{n}\times\mathbf{e}\delta(\mathbf{y}-\mathbf{y}_w)$, where the δ-function is a two-dimensional one in the **n**, **e** plane. Then one finds

$$\mathbf{y}\times\mathbf{w} = \Gamma\{(\mathbf{a}.\mathbf{e})\,\mathbf{n}-(\mathbf{a}.\mathbf{n})\,\mathbf{e}+R\mathbf{n}\}\,\delta(\mathbf{y}-\mathbf{y}_w).$$

If one uses $\int \mathbf{ee}\delta(\mathbf{y}-\mathbf{y}_w)\,d^3y = R(\mathbf{I}-\mathbf{nn})$, where **I** denotes the unit tensor, one obtains from (17)

$$p = \frac{\rho_0}{4a_0^2x^3}\frac{\partial^3}{\partial t^3}\Gamma R^2\mathbf{x}.(\mathbf{an}-\tfrac{1}{3}\mathbf{a}.\mathbf{n}).\mathbf{x}.$$

If the ring moves in the direction of its normal one finds with $\mathbf{a} = \xi\mathbf{n}$

$$p = \frac{\rho_0}{4a_0^2x^3}\frac{d^3\Gamma R^2\xi}{dt^3}\mathbf{x}.(\mathbf{nn}-\tfrac{1}{3}).\mathbf{x}.$$

If there are several vortex rings one has just to add their contributions. For a system of circular coaxial rings, the generated sound is related to the third derivative of the mean axial position of the vortex system (Lamb 1948, art. 162).

For two identical vortex rings with circulation Γ and ring diameters R_1 and R_2 at positions ξ_1 and ξ_2 one finds

$$p = \frac{\rho_0}{4a_0^2x^3}\frac{d^3s}{dt^3}\mathbf{x}.(\mathbf{nn}-\tfrac{1}{3}\}.\mathbf{x}, \quad s = \Gamma(R_1^2\xi_1+R_2^2\xi_2).$$

Lamb shows that ds/dt is related to the energy T of the vortex system by

$$\frac{ds}{dt} = \frac{T}{2\pi\rho_0}+3\Gamma(R_1\dot{R}_1\xi_1+R_2\dot{R}_2\xi_2) = \frac{T}{2\pi\rho_0}+3\Gamma R_1\dot{R}_1(\xi_1-\xi_2).$$

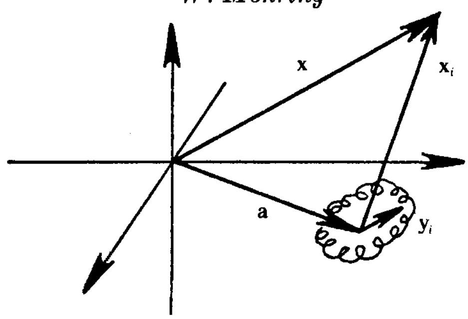

FIGURE 3. The configuration of the ith vorticity spot.

If one considers very thin rings with a separation d small compared with their diameter R_0 and large compared with their core diameter, one may identify their relative motion with that of a pair of rectilinear spinning vortices and assume

$$\xi_1 = u_0 t + \frac{d}{2}\cos\frac{\Gamma}{\pi d^2}t, \quad R_1 = R_0 + \frac{d}{2}\sin\frac{\Gamma}{\pi d^2}t,$$

$$\xi_2 = u_0 t - \frac{d}{2}\cos\frac{\Gamma}{\pi d^2}t, \quad R_2 = R_0 - \frac{d}{2}\sin\frac{\Gamma}{\pi d^2}t$$

to hold, where u_0 is the translation velocity of the vortex system. (The same result can be obtained by lengthy calculations from Hicks' (1923) results.) This leads to

$$\frac{ds}{dt} = \frac{T}{2\pi\rho_0} + \frac{3}{4}\frac{\Gamma^2}{\pi}R_0\left(1+\cos\frac{2\Gamma}{\pi d^2}t\right)$$

and to

$$p = -\frac{3}{4}\frac{\rho_0\Gamma^4 R_0}{\pi^3 a_0^2 d^4 x^3}\cos\frac{2\Gamma}{\pi d^2}\left(t-\frac{x}{a_0}\right)\mathbf{x}.(\mathbf{nn}-\tfrac{1}{3}).\mathbf{x}.$$

The transformation of (9) can also be applied to a flow field which is generated by a number of compact vorticity spots. Then (8) can be written as a sum over these spots:

$$p(\mathbf{x},t) = -\rho_0 \sum_i \int_{V_i} \nabla_y G(\mathbf{x},\mathbf{y},t-t').\mathbf{L}(\mathbf{y},t')\,d^3y\,dt', \tag{8a}$$

where V_i denotes the volume of the ith spot (figure 3). If it is assumed that the approximation (15) is valid in every spot one finds

$$\nabla_y G = \frac{1}{4\pi x}\left[\nabla_{y_i}\times\left\{\frac{1}{2}\frac{\mathbf{x}\times\mathbf{y}_i}{x a_0}\delta'(t_i^r)+\frac{1}{3}\frac{(\mathbf{x}.\mathbf{y}_i)\,\mathbf{x}\times\mathbf{y}_i}{x^2a_0^2}\delta''(t_i^r)\right\}+\frac{1}{3a_0^2}\mathbf{y}_i\,\delta''(t_i^r)\right],$$

$$t_i^r = t - t' - x_i/a_0,$$

to be valid in the ith spot. Inserting this into (8a) one obtains for the generated sound

$$p(\mathbf{x},t) = \frac{1}{4\pi x}\sum_i\left[-\frac{\mathbf{x}}{a_0 x}.\frac{d^2\mathbf{P}_i(t_i^r)}{dt^2}+\frac{\mathbf{x}}{a_0 x}.\frac{d^2\mathbf{Q}_i(t_i^r)}{dt^2}.\frac{\mathbf{x}}{a_0 x}-\frac{1}{3}\frac{d^2T_i(t_i^r)}{dt^2}\right],$$

where

$$\mathbf{P}_i = \tfrac{1}{2}\rho_0\int_{V_i}\mathbf{y}_i\times\mathbf{w}\,d^3y_i$$

denotes the impulse of the ith vortex,

$$\mathbf{Q}_i = \tfrac{1}{3}\rho_0\frac{d}{dt}\int_{V_i}\mathbf{y}_i(\mathbf{y}_i\times\mathbf{w})\,d^3y_i$$

is the tensor associated with the ith vortex found already in (17) and

$$T_i = \rho_0 \int_{V_i} \mathbf{u}.(\mathbf{y}_i \times \mathbf{w})\, d^3y_i$$

is the energy of the ith vortex (Lamb 1948, art. 152). In this approximation each vortex acts acoustically like a dipole with a moment equal to the time derivative of the vortex impulse, plus a quadrupole with moment $\mathbf{Q}_i$ plus a spherically symmetric source with strength equal to $-\frac{1}{3}$ of the second derivative of the vortex energy. In the case of a single vorticity spot, this result reduces to (17) because the vortex impulse and energy are then constant.

This scheme can also be applied to the sound generated by turbulence. Then one has to evaluate the average value of (6). With the Green's function (14) one finds for the Green's function product in (6)

$$GG' = \frac{1}{16\pi^2 x^2}\,\delta\left(t - t' - \frac{x}{a_0} + \frac{\mathbf{x}.\mathbf{y}}{xa_0}\right)\delta\left(\tau - \frac{\mathbf{x}.(\mathbf{y}-\mathbf{y}')}{xa_0}\right), \qquad \tau = t' - t''.$$

If one expands the second δ-function and uses the fact that to lowest order only terms which are quadratic in $\mathbf{y}$ and $\mathbf{y}'$ lead to non-vanishing contributions, GG' simplifies to

$$GG' = \frac{1}{64\pi^2 a_0^4 x^5}\,\delta\left(t - t' - \frac{x}{a_0} + \frac{\mathbf{x}.\mathbf{y}}{xa_0}\right)\delta^{\mathrm{iv}}(\tau)\,(\mathbf{x}.\mathbf{y})^2\,(\mathbf{x}.\mathbf{y}')^2.$$

If this is inserted in (6) together with

$$\int (\mathbf{x}.\mathbf{y})^2 \operatorname{div} \mathbf{L}\, d^3y = \frac{2}{3}\int (\mathbf{x}.\mathbf{y})\,\mathbf{x}\times\mathbf{y}.\frac{\partial \mathbf{w}}{\partial t}\, d^3y$$

one finds for the average intensity

$$\langle I\rangle = -\frac{\rho_0}{144\pi^2 x^6 a_0^5}\iint (\mathbf{x}.\mathbf{y})\,(\mathbf{x}.\mathbf{y}')\,\mathbf{x}\times\mathbf{y}.\frac{\partial^6}{\partial\tau^6}\langle \mathbf{w}(\mathbf{y},t')\,\mathbf{w}(\mathbf{y}',t'-\tau)\rangle.\mathbf{x}\times\mathbf{y}'\, d^3y\, d^3y_{\tau=0}.$$

This expression relates the generated sound linearly to the vorticity correlation tensor, which is linearly related to the velocity correlation tensor.

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